The Likelihood Function for Stochastic Volatility Models

Let

$$\begin{aligned} r_k &:= \mathrm{ret}(t_k) \\ v_k &:= \mathrm{vol}(t_k) \end{aligned}$$

denote the returns and the instantaneous volatility for the models

$$S(t_k) \;=\; S(t_{k-1}) \times \big[1 + \mathrm{vol}(t_{k-1})\,\phi_k\big] \tag{1}$$

For the ARCH and GARCH models, where the volatility is completely determined by the return history, the likelihood function is given by

$$\begin{aligned} L &= \prod_{k=2}^{N} \frac{1}{[2\pi v_{k-1}^2]^{1/2}}\, \exp\Big\{-\frac{r_k^2}{2v_{k-1}^2}\Big\} \\ &= \prod_{k=1}^{N-1} \frac{1}{[2\pi v_k^2]^{1/2}}\, \exp\Big\{-\frac{r_{k+1}^2}{2v_k^2}\Big\} \end{aligned} \tag{2}$$

For a stochastic volatility model, there is a second set of random numbers $(\varepsilon_k)_{k=1}^N$, also taken from a standard normal distribution, and the instantaneous volatility at time k can be computed from the vol at time $k-1$ and the realization of ε_k,

$$v_k \;=\; v_k(\varepsilon_k, v_{k-1}) \tag{3}$$

For example, for the GARCH-Diffusion model we have

$$v_k^2 \;=\; v_{k-1}^2 \;+\; \kappa(\mathrm{bsvol}^2 - v_{k-1}^2) \;+\; \beta v_{k-1}^2\,\varepsilon_k \tag{4}$$

Or we may simply consider a test model with volatility

$$v_k^2 \;=\; \sigma^2 \varepsilon_k^2 \tag{5}$$

in which case $v_k = v_k(\varepsilon_k)$ is actually independent of v_{k-1}. In both cases (4) and (5) or for the general case (3), the likelihood function is given by the following expression:

$$L \;=\; \int_{\mathbb{R}^{N-1}} \prod_{k=1}^{N-1} \frac{1}{[2\pi v_k^2]^{1/2}}\, \exp\Big\{-\frac{r_{k+1}^2}{2v_k^2}\Big\}\; e^{-\frac{\varepsilon_1^2+\cdots\varepsilon_{N-1}^2}{2}}\, \frac{d^{N-1}\varepsilon}{(2\pi)^{(N-1)/2}} \tag{6}$$

$$= \int_{\mathbb{R}^{N-1}} \prod_{k=1}^{N-1} \frac{1}{[2\pi v_k(\varepsilon_k, v_{k-1})^2]^{1/2}} \exp\{-\frac{r_{k+1}^2}{2v_k(\varepsilon_k, v_{k-1})^2}\} \; e^{-\frac{\varepsilon_1^2+\cdots\varepsilon_{N-1}^2}{2}} \frac{d^{N-1}\varepsilon}{(2\pi)^{(N-1)/2}}$$

Usually, the integrals in (6) cannot be performed in closed form and some kind of numerical algorithm has to be put in place in order to calculate L. That we will do next week with a Monte Carlo type algorithm. For now, we will consider a demo-test-model where the likelihood function and its maximum can be analytically calculated. This will then also serve as a test-model for the algorithm which we will implement next week.

An Explicit Solution for the Likelihood-Function for a Demo-Test-Model:

Instead of the model (5), we consider the following model which is actually easier to solve:

$$v_k^2 \;=\; \sigma^2 \, \frac{\varepsilon_{1,k}^2 + \varepsilon_{2,k}^2}{2} \tag{7}$$

Furthermore, let us assume for notational simplicity that the time series under consideration has length $N+1$ and not just N, then we can substitute all the $N-1$'s in (6) by N's which shortens the formulae a bit:

$$L \;=\; \int_{\mathbb{R}^{N}} \prod_{k=1}^{N} \frac{1}{[2\pi v_k(\varepsilon_k, v_{k-1})^2]^{1/2}} \exp\{-\frac{r_{k+1}^2}{2v_k(\varepsilon_k, v_{k-1})^2}\} \; e^{-\frac{\varepsilon_1^2+\cdots\varepsilon_{N}^2}{2}} \frac{d^{N}\varepsilon}{(2\pi)^{N/2}} \tag{8}$$

Since for the specification (7) there are now two standard normally distributed random numbers $(\varepsilon_{1,k}, \varepsilon_{2,k})$ for each k instead of just one single ε_k,, the likelihood function (8) above changes to

$$L \;=\; \int_{\mathbb{R}^{2N}} \prod_{k=1}^{2N} \frac{1}{[2\pi v_k(\varepsilon_{1,k}, \varepsilon_{2,k})^2]^{1/2}} \exp\{-\frac{r_{k+1}^2}{2v_k(\varepsilon_{1,k}, \varepsilon_{2,k})^2}\} \; e^{-\frac{\varepsilon_{1,1}^2+\varepsilon_{2,1}^2+\cdots\varepsilon_{1,N}^2+\varepsilon_{2,N}^2}{2}} \frac{d^{N}\varepsilon_1 \, d^{N}\varepsilon_2}{(2\pi)^{N}} \tag{9}$$

Now we calculate as follows: First, we use the following identity:

$$\int_{\mathbb{R}} e^{-irx} \, e^{-v^2 \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \;=\; \frac{1}{\sqrt{v^2}} \, e^{-\frac{1}{v^2}\frac{r^2}{2}} \tag{10}$$

If one substitutes $x \to y = vx$ in (10), it reads

$$\int_{\mathbb{R}} e^{-i\frac{r}{v}y} \, e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \;=\; e^{-\frac{1}{2}(\frac{r}{v})^2} \tag{11}$$

and we will check this in the exercises with a simple Monte Carlo simulation (also in order to refresh simple Monte Carlo at this place which will also show up in the Klausur).

Then, using (10), we can rewrite the exponentials in (9) above as follows:

$$\begin{aligned} \frac{1}{[2\pi v_k^2]^{1/2}} \exp\{-\frac{r_{k+1}^2}{2v_k^2}\} \;&=\; \int_{\mathbb{R}} e^{-ir_{k+1}x} \, e^{-v_k^2\frac{x^2}{2}} \frac{dx}{2\pi} \\ &=\; \int_{\mathbb{R}} e^{-ir_{k+1}x_k} \, e^{-v_k^2\frac{x_k^2}{2}} \frac{dx_k}{2\pi} \end{aligned} \tag{12}$$

Thus we obtain:

$$L \;=\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^{2N}} \exp\Big\{-\sum_{k=1}^{N}\big[ir_{k+1}x_k + v_k(\varepsilon_{1,k},\varepsilon_{2,k})^2\,\frac{x_k^2}{2}\big]\Big\}\; e^{-\frac{\varepsilon_{1,1}^2+\varepsilon_{2,1}^2+\cdots\varepsilon_{1,N}^2+\varepsilon_{2,N}^2}{2}}\,\frac{d^N\varepsilon_1\,d^N\varepsilon_2}{(2\pi)^N}\;\;\frac{d^Nx}{(2\pi)^N}$$

Now we substitute the vol-specification (5),

$$v_k^2 \;=\; \sigma^2\,\frac{\varepsilon_{1,k}^2+\varepsilon_{2,k}^2}{2}$$

and obtain

$$\begin{aligned} L \;&=\; \int_{\mathbb{R}^N}\frac{d^Nx}{(2\pi)^N}\,\exp\Big\{-\sum_{k=1}^{N}\big[ir_{k+1}x_k\big]\Big\}\times \\ &\qquad \prod_{k=1}^{N}\int_{\mathbb{R}^2}\exp\Big\{-\sigma^2\,\frac{\varepsilon_{1,k}^2+\varepsilon_{2,k}^2}{2}\,\frac{x_k^2}{2}\Big\}\exp\Big\{-\frac{\varepsilon_{1,k}^2+\varepsilon_{2,k}^2}{2}\Big\}\,\frac{d\varepsilon_{1,k}d\varepsilon_{2,k}}{2\pi} \\ &=\; \int_{\mathbb{R}^N}\frac{d^Nx}{(2\pi)^N}\,\exp\Big\{-\sum_{k=1}^{N}\big[ir_{k+1}x_k\big]\Big\}\prod_{k=1}^{N}\int_{\mathbb{R}^2}\exp\Big\{-\big[1+\frac{\sigma^2x_k^2}{2}\big]\frac{\varepsilon_{1,k}^2+\varepsilon_{2,k}^2}{2}\Big\}\,\frac{d\varepsilon_{1,k}d\varepsilon_{2,k}}{2\pi} \end{aligned} \tag{13}$$

The 2-dimensional integral in the last line of (13) becomes

$$\int_{\mathbb{R}^2}\exp\Big\{-\big[1+\frac{\sigma^2x_k^2}{2}\big]\frac{\varepsilon_{1,k}^2+\varepsilon_{2,k}^2}{2}\Big\}\,\frac{d\varepsilon_{1,k}d\varepsilon_{2,k}}{2\pi} \;=\; \frac{1}{1+\frac{\sigma^2x_k^2}{2}} \tag{14}$$

and we arrive at

$$\begin{aligned} L \;&=\; \int_{\mathbb{R}^N}\frac{d^Nx}{(2\pi)^N}\,\exp\Big\{-\sum_{k=1}^{N}\big[ir_{k+1}x_k\big]\Big\}\prod_{k=1}^{N}\frac{1}{1+\frac{\sigma^2x_k^2}{2}} \\ &=\; \prod_{k=1}^{N}\Big\{\int_{\mathbb{R}}\frac{dx_k}{2\pi}\,e^{-ir_{k+1}x_k}\,\frac{1}{1+\frac{\sigma^2x_k^2}{2}}\Big\} \\ &=\; \prod_{k=1}^{N}\Big\{\frac{\sqrt{2}}{\sigma}\int_{\mathbb{R}}\frac{dy_k}{2\pi}\,e^{-i\sqrt{2}\frac{r_{k+1}y_k}{\sigma}}\,\frac{1}{1+y_k^2}\Big\} \end{aligned} \tag{15}$$

Now we use another identity:

$$\tfrac{1}{2}\,e^{-|q|} \;=\; \int_{\mathbb{R}}\frac{dy}{2\pi}\,e^{-iqy}\,\frac{1}{1+y^2} \tag{16}$$

which we will also check in the exercises. Substituting (16) in (15), we arrive at the compact expression

$$L \;=\; \prod_{k=1}^{N}\Big\{\frac{\sqrt{2}}{\sigma}\,\frac{1}{2}\,\exp\Big(-\frac{\sqrt{2}}{\sigma}|r_{k+1}|\Big)\Big\} \tag{17}$$

or

$$\log L \;=\; \log L(\sigma) \;=\; -N\log[\sigma\sqrt{2}] \;-\; \frac{\sqrt{2}}{\sigma}\sum_{k=1}^{N}|r_{k+1}|$$

which gives

$$\frac{d\log L(\sigma)}{d\sigma} \;=\; -\frac{N}{\sigma} \;+\; \frac{\sqrt{2}}{\sigma^2}\sum_{k=1}^{N}|r_{k+1}|$$

and we obtain the Maximum Likelihood estimator

$$\hat{\sigma}_{MLE} \;=\; \frac{\sqrt{2}}{N}\sum_{k=1}^{N}|r_{k+1}| \quad . \tag{18}$$

The value of the likelihood function at the maximum is given by

$$\begin{aligned} L(\hat{\sigma}_{MLE}) &= \frac{1}{(\hat{\sigma}_{MLE}\sqrt{2})^N}\;\exp\Big\{-\frac{\sqrt{2}}{\hat{\sigma}_{MLE}}\sum_{k=1}^{N}|r_{k+1}|\Big\} \\ &= \frac{1}{\big\{\,\frac{2}{N}\sum_{k=1}^{N}|r_{k+1}|\,\big\}^N}\;e^{-N} \\ &= \frac{1}{\big\{\,\frac{2e}{N}\sum_{k=1}^{N}|r_{k+1}|\,\big\}^N} \end{aligned} \tag{19}$$

or

$$\log L(\hat{\sigma}_{MLE}) \;=\; -N\log\Big[\frac{2e}{N}\sum_{k=1}^{N}|r_{k+1}|\,\Big] \tag{20}$$